On the Nusselt number for thermally developed flow between isothermal parallel plates with dissipation

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ROHSENOW KENDALL HEAT TRANSFER LABORATORY

A startling claim about Nu!

"In all common heat transfer books, [Nu] for laminar fully developed incompressible flow in a plane channel with isothermal walls is Nu = 7.54 ...However, when the viscous dissipation is very small but not identically zero (which is a physical impossibility), ...Nu = 35/2 for a liquid or Nu = 0 for an ideal gas." [1]

Ref. [1] states that most heat transfer textbooks quote "*wrong results*" and that dissipative solutions have been "*not understood, overlooked, and forgotten.*"

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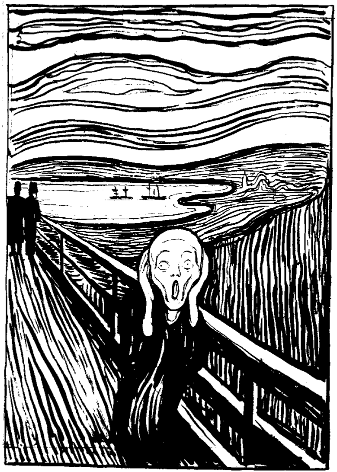

Edvard Munch, 1895

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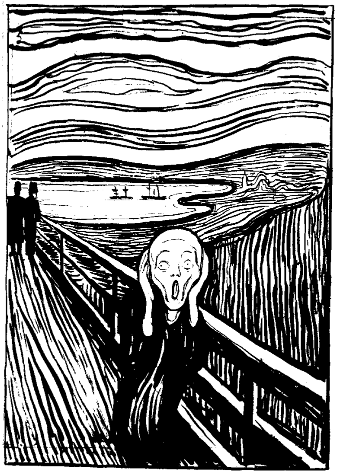

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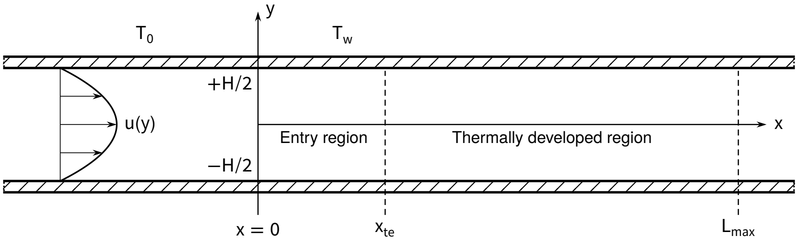
A similar result had been reported in 1973 [2], but not adopted.

Fully developed laminar flow between parallel plates



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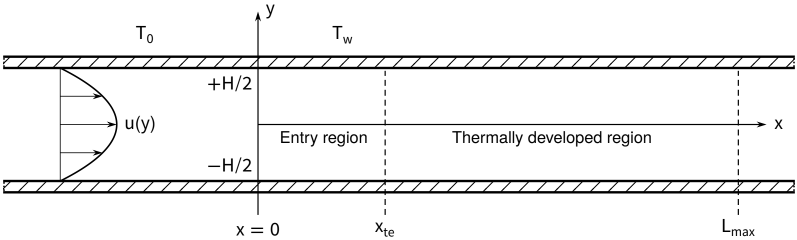

Fully-developed flow

$$u(y) = \frac{3}{2} u_b \left[1 - 4 \left(\frac{y}{H} \right)^2 \right]$$

$$-\frac{dp}{dx} = \frac{12 \mu u_b}{H^2}$$

Thermal energy equation ($dh = T ds + v\, dp$)

$$\rho c_p \, \mathbf{u} \cdot \nabla T = -\nabla \cdot \mathbf{q} + \beta T \, \mathbf{u} \cdot \nabla p + d_{ji} \frac{\partial u_i}{\partial x_j}$$

with $u(y)$ and neglecting axial conduction

$$\rho c_p u(y) \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + \underbrace{\beta T u(y) \frac{dp}{dx} + \mu \left(\frac{du}{dy} \right)^2}_{\text{fn}(y \text{ only})}$$

Solution by superposition

Separate the contribution of flow work and dissipation

$$T(x,y) = T_{\text{pp}}(x,y) + T_{\text{diss}}(y)$$

T_{pp} is just the classical Graetz problem

$$\rho c_p u(y) \frac{\partial T_{\text{pp}}}{\partial x} = k \frac{\partial^2 T_{\text{pp}}}{\partial y^2}, \qquad T_{\text{pp}} = \begin{cases} T_w & x \geq 0, y = \pm H/2 \\ T_0 & x = 0, \forall y \end{cases}$$

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The second is problem is an ode (dissipation & flow work $\neq \text{fn}(x)$):

$$0 = k \frac{d^2 T_{\text{diss}}}{dy^2} + \underbrace{(\beta T)}_{\simeq \text{const.}} u(y) \frac{dp}{dx} + \mu \left(\frac{du}{dy} \right)^2, \qquad T_{\text{diss}}(y = \pm H/2) = 0$$

NB: $\beta T = 0$ if incompressible, $\beta T = 1$ for ideal gas.

Graetz solution (no flow work or dissipation)

The classical result for step-change wall temperature is [3–9]

$$
\begin{aligned}
\theta_{\text{pp}}(x^+, \bar{y}) &= \frac{T_{\text{pp}} - T_w}{T_0 - T_w} \\
&= B_0 \exp\left(-\frac{32}{3}\lambda_0^2 x^+\right) Y_0(\bar{y}) + \sum_{n=1}^{\infty} B_n \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) Y_n(\bar{y})
\end{aligned}
$$

$$\text{for } x^+ = \frac{(x/D_h)}{\text{Re}_{D_h}\text{Pr}} \text{ and } \bar{y} = \frac{y}{H}$$

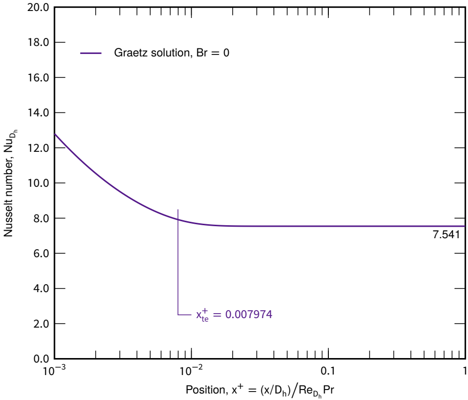
with bulk temperature

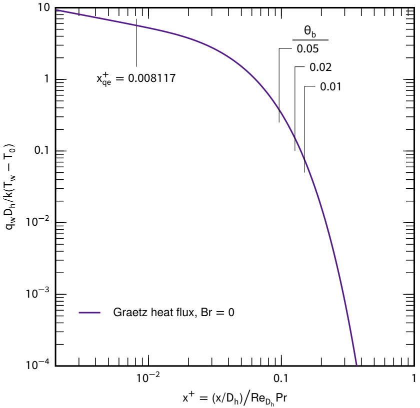
$$\theta^b_{\text{pp}}(x^+) = 3\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right)$$

and wall heat flux (at $\bar{y} = 1/2$) is

$$q_{w,\text{pp}}(x^+) = \frac{8(T_0 - T_w)k}{D_h} \sum_{n=0}^{\infty} G_n \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right)$$

Graetz solution: Dimensionless flux & Nusselt number





x^+_{te} : Nu_{D_h} within 5% of thermally developed value (one-term solution)

x^+_{qe} : q_w within 5% of thermally developed value (one-term solution)

Practical heat exchangers stop at a finite pinch, with $\theta^b > 0$

Contribution of dissipation & flow work

$$\theta_{\text{diss}} = \frac{T_{\text{diss}}}{\Delta T} = \text{Br}\underbrace{\left[9(\beta T)\left(\bar{y}^2 - \frac{2}{3}\bar{y}^4\right) - 12\bar{y}^4 - \frac{15}{8}(\beta T) + \frac{3}{4}\right]}_{=\text{fn}(\bar{y})}$$

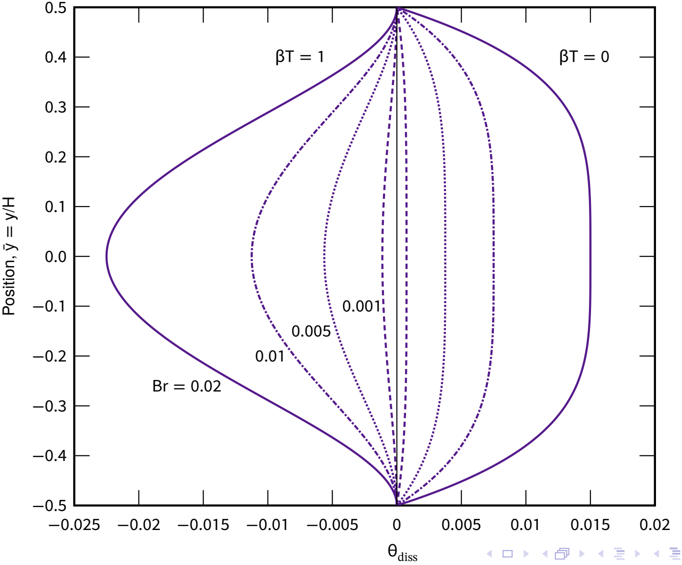
with $\text{Br} = \mu u_b^2/k\Delta T,$ and $\Delta T = T_w - T_0$. Bulk temperature rise is

$$\theta^b_{\text{diss}} = a\,\text{Br} = \begin{cases} +\dfrac{24}{35}\,\text{Br} & \beta T = 0,\ \text{incompressible liquid} \\ -\dfrac{27}{35}\,\text{Br} & \beta T = 1,\ \text{ideal gas} \end{cases}$$

and wall temperature gradient is

$$\left.\frac{d\theta_{\text{diss}}}{d\bar{y}}\right|_{\bar{y}=+H/2} = 6\text{Br}\,[\beta T - 1] = \begin{cases} -6\text{Br} & \text{incompressible liquid} \\ 0 & \text{ideal gas !!} \end{cases}$$

Dissipative temperature rise for ideal gases and incompressible liquids



High values of the Brinkman number ($\mathrm{Br} = \mu u_b^2/k\Delta T$)

Consider $\Delta T = T_0 - T_w = 10$ K and mean temperatures of about 310 K.

Air: at $u_b = 15$ m/s, $H = 1$ mm: $\mathrm{Re}_{D_h} = 1797$, $\mathrm{Br} = 0.0158$

$T^b_{\mathrm{diss}} = -(27/35)\mathrm{Br}\Delta T = -0.122$ K

Water: at $u_b = 0.7$ m/s, $H = 1$ mm: $\mathrm{Re}_{D_h} = 2005$, $\mathrm{Br} = 5.43 \times 10^{-5}$

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Glycerol: at $u_b = 0.5$ m/s, $H = 5$ mm: $\mathrm{Re}_{D_h} = 22$, $\mathrm{Br} = 0.0249$

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-0.7 *atm/m pressure gradient...getting large*

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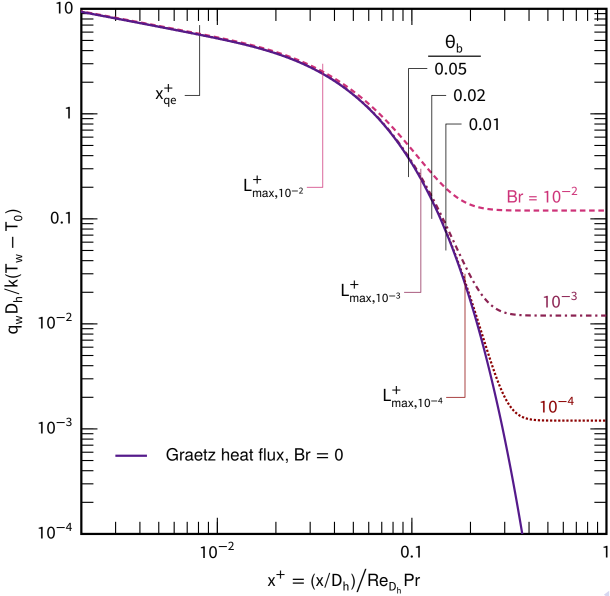
Raising u_b to 5 m/s gives: $\mathrm{Re}_{D_h} = 220$, $\boxed{\mathrm{Br} = 2.49}$, $T^b_{\text{diss}} = 17.1$ K, and

$dp/dx = -7$ atm/m...

Dimensionless heat flux in liquids: $q_w = q_{w,\mathrm{pp}} + q_{w,\mathrm{diss}}$



$$q_{w,\mathrm{diss}}/q_w \leqslant 5\%$$
$$\text{for } x^+ \leqslant L^+_{\max,\mathrm{Br}}$$

So what about the Nusselt number?

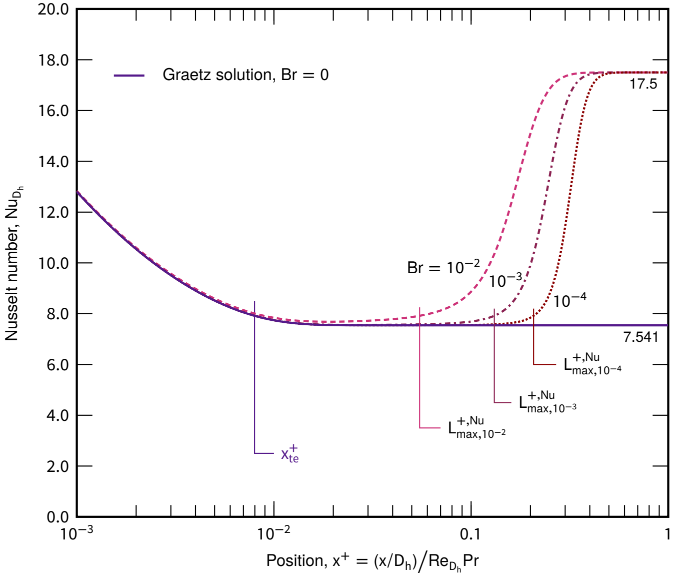
Naively using $h = q_w/(T_b - T_w)$, as in Refs. [1,2], gives

$$\mathrm{Nu}_{D_h} = \frac{q_w D_h}{k(T_0 - T_w)\theta^b} = \frac{8\sum_{n=0}^{\infty} G_n \exp\left[-(32/3)\lambda_n^2 x^+\right] + 12\mathrm{Br}}{3\sum_{n=0}^{\infty}(G_n/\lambda_n^2)\exp\left[-(32/3)\lambda_n^2 x^+\right] + a\mathrm{Br}}$$

where blue-colored term is only for liquids.

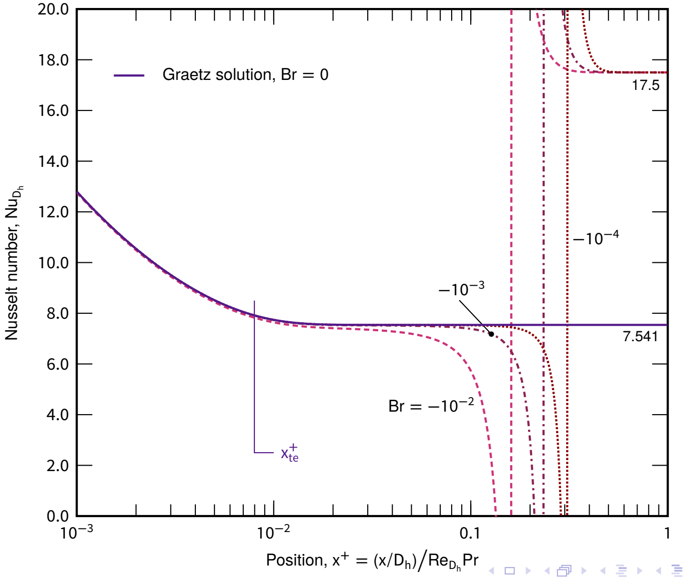
The result behaves strangely as $x^+ \longrightarrow \infty$ ...

Nu without separating dissipation (liquids, $\mathrm{Br} > 0$)



Nu without separating dissipation (liquids, $\mathrm{Br} < 0$)

For $T_w > T_0$ (heating), $\mathrm{Br} < 0$



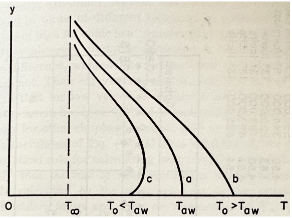
Superposition in high-speed flow and channels

Dissipation in a high-speed boundary layer heats an adiabatic wall. Heat is rejected to free stream.

For an isothermal wall, a superposition solution shows

$$q_w = h(T_{aw} - T_w)$$

with h for flow w/o dissipation [8,12].



Rohsenow & Choi, 1961

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Superposition gives us

$$\begin{aligned} q_w &= q_{\text{pp}} + q^{\text{liq}}_{\text{diss}} \\ &= h_{\text{pp}}(T^b_{\text{pp}} - T_w) + q^{\text{liq}}_{\text{diss}} \end{aligned}$$

where h_{pp} is the Graetz value.

We can just compute the heat transfer using classical Graetz, and then add back $q^{\text{liq}}_{\text{diss}}$ (if it's not negligible)

Similarly, $T_b = T_{\text{pp}} + T^b_{\text{diss}}$.

Example: the glycerol flow mentioned earlier (0.5 m/s)

$$T_0 = 318 \text{ K}$$
$$T_w = 308 \text{ K}$$
$$x = 2.5 \text{ m}$$
$$x^+ = 0.00464$$

$$\text{Nu}_{\text{Graetz}} = 7.541$$
$$h_{\text{pp}} = 214.9 \text{ W/m}^2\text{K}$$

$$T^b_{\text{pp}} = 315.92 \text{ K}$$
$$q_{w,\text{pp}} = 214.9(315.92 - 308) = 1701 \text{ W/m}^2$$

Adding on the dissipative increments,

$$T^b_{\text{diss}} = 0.171 \text{ K}$$
$$q^{\text{liq}}_{\text{diss}} = 85.06 \text{ W/m}^2 \text{ (NB: +5\%)}$$

we find $T_b = 316.09$ K and $q_w = 1786$ W/m^2.

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The same results are obtained using the strange Nu that includes dissipation. But with superposition, we *don't need* a complicated additional Nusselt number; and thermally developed flow is as we have been teaching it.

Seban-Shimazaki criterion: 1949 [10], 1951 [11]

$$\text{For thermally developed flow:} \qquad \frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = 0$$

This criterion is a short-cut for analysis of thermally developed flow, ensures h independent of x.

Seban-Shimazaki criterion: 1949 [10], 1951 [11]

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One-term limit of Graetz loses x dependence (S-S crit. met automatically):

$$\left(\frac{T_w - T_{\mathrm{pp}}}{T_w - T_b}\right) = \left[\frac{B_0 \exp\left(-32\lambda_0^2 x^+/3\right) Y_0(\bar{y}) + \sum_{n=1}^{\infty} B_n \exp\left(-32\lambda_n^2 x^+/3\right) Y_n(\bar{y})}{3(G_0/\lambda_0^2)\exp\left(-32\lambda_0^2 x^+/3\right) + 3\sum_{n=1}^{\infty}(G_n/\lambda_n^2)\exp\left(-32\lambda_n^2 x^+/3\right)}\right]$$

$$\simeq \left(\frac{\lambda_0^2 B_0 Y_0(\bar{y})}{3G_0}\right) \text{ for } x^+ > x_{te}^+$$

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$$\simeq \left(\frac{\lambda_0^2 B_0 Y_0(\bar{y})}{3G_0}\right) \quad \text{for } x^+ > x_{te}^+$$

One-term limit with flow work & dissipation depends on x^+ until transient completely decays:

$$\left(\frac{T_w - T}{T_w - T_b}\right) \simeq \left(\frac{B_0 \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) Y_0(\bar{y}) + \mathrm{Br}\,\mathrm{fn}(\bar{y})}{3(G_0/\lambda_0^2) \exp\left(-\frac{32}{3}\lambda_n^2 x^+\right) + a\mathrm{Br}}\right)$$

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Ref. [1] used the criterion with an h that included dissipation and flow work: that led to a confused result.

Summary

1. Dissipation in channels is an old problem (and sometimes a homework problem...)

J. H. Lienhard, “On the Nusselt number for thermally developed flow between isothermal parallel plates with dissipation,” *ASME J. Heat Mass Transfer*, June 2025, https://doi.org/10.1115/1.4068967 (open access)

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6. **The textbook literature states the correct values of the thermally developed Nusselt number**

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Additional Slides & References

Criterion to neglect dissipation effect on q_w (liquids)

One-term Graetz solution including dissipation (thermally developed region)

$$q_w(x^+) \simeq \frac{(T_0 - T_w)k}{D_h}\left[8G_0 \exp\left(-32\lambda_0^2 x^+/3\right) + 12\mathrm{Br}\right]$$

The one-term Graetz solution *neglecting dissipation* is within 5% of the complete solution for

$$0.008117 \le x^+ < -0.1178 - 0.03315 \ln|\mathrm{Br}| \equiv L^+_{\max}$$

Eigenvalues and coefficients for Graetz solution [9]

Table 1: Eigenvalues and coefficients for Graetz solution [9]. For higher n: $\lambda_n \doteq 4n + 5/3$ and $G_n \doteq 1.01278\,\lambda_n^{-1/3}$ [4].

n	λ_n	G_n
0	1.68159	0.858087
1	5.66985	0.569463
2	9.66842	0.476065

Brinkman number at which liquid dissipation makes less than a 5% or 1% contribution to θ^b or q_w

Table 2: Brinkman number at which liquid dissipation makes less than a 5% or 1% contribution to θ^b or q_w from the Graetz solution at the points where θ^b equals 0.05, 0.02, or 0.01.

			$\lvert\mathrm{Br}\rvert$			
θ^b	x^+	$q_w D_h / k\Delta T$	5% of θ^b	1% of θ^b	5% of q_w	1% of q_w
0.05	0.09621	0.3770	3.646×10^{-3}	7.292×10^{-4}	1.571×10^{-3}	3.142×10^{-4}
0.02	0.1266	0.1508	1.458×10^{-3}	2.917×10^{-4}	6.824×10^{-4}	1.257×10^{-4}
0.01	0.1496	0.07541	7.292×10^{-4}	1.458×10^{-4}	3.142×10^{-4}	6.284×10^{-5}

Why do flow work and dissipation cancel for ideal gases?

At the local level, they don't: some dissipated mechanical energy is conducted as heat from the high-shear region near the wall to the interior.

At the bulk level, the work dissipated to drive the gas against viscosity balances the flow work released by the pressure gradient.

This can be shown by constructing the usual force balance (between wall friction and pressure drop) for steady flow and multiplying by u_b to get power. The result simply shows that the rate of dissipating mechanical energy by friction equals the flow work done.

For an incompressible liquid with $\beta T = 0$, the flow work does not appear in the enthalpy equation, but dissipation still does.

Are gas expansion effects larger than dissipation?

In gases, pressure drop cooling and dissipative heating combine to produce zero wall heat flux: dissipated work exactly offsets expansion in the bulk sense. The expansion is *not* adiabatic b/c volumetric heat source: pressure drops isothermally.

The pressure gradient in the high Br airflow is about −3300 Pa/m. Bulk density drops by 3.3% per meter. Kinetic energy will rise by 3.3% per meter (but at 15 m/s, Ma $<$0.05).

Of course, heating and cooling may change the state of the gas by much more!

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